9.4 An electromagnetic wave

We are going to construct a rather simple electromagnetic field that will satisfy Maxwell's equations for empty space, Eq. (9.18). Suppose there is an electric field **E**, everywhere parallel to the z axis, whose intensity depends only on the space coordinate y and the time t. Let the dependence have this particular form:[1]

$$\mathbf{E} = \hat{\mathbf{z}} E_0 \sin(y - vt), \tag{9.22}$$

in which E_0 and v are simply constants. This field fills all space – at least all the space we are presently concerned with. We'll need a magnetic field, too. We shall assume that it has an x component only, with a dependence on y and t similar to that of E_z:

$$\mathbf{B} = \hat{\mathbf{x}} B_0 \sin(y - vt), \tag{9.23}$$

where B_0 is another constant.

Figure 9.7 may help you to visualize these fields. It is difficult to represent graphically two such fields filling all space. Remember that nothing varies with x or z; whatever is happening at a point on the y axis is happening everywhere on the perpendicular plane through that point. As time goes on, the entire field pattern slides steadily to the right,

[1] There is technically an issue with the units here, because the argument of the sine function should be dimensionless. We should really be writing it as $\sin(ky - \omega t)$ or something similar; see the example in Section 9.5. However, the present form makes things a little less cluttered, without affecting the final results.

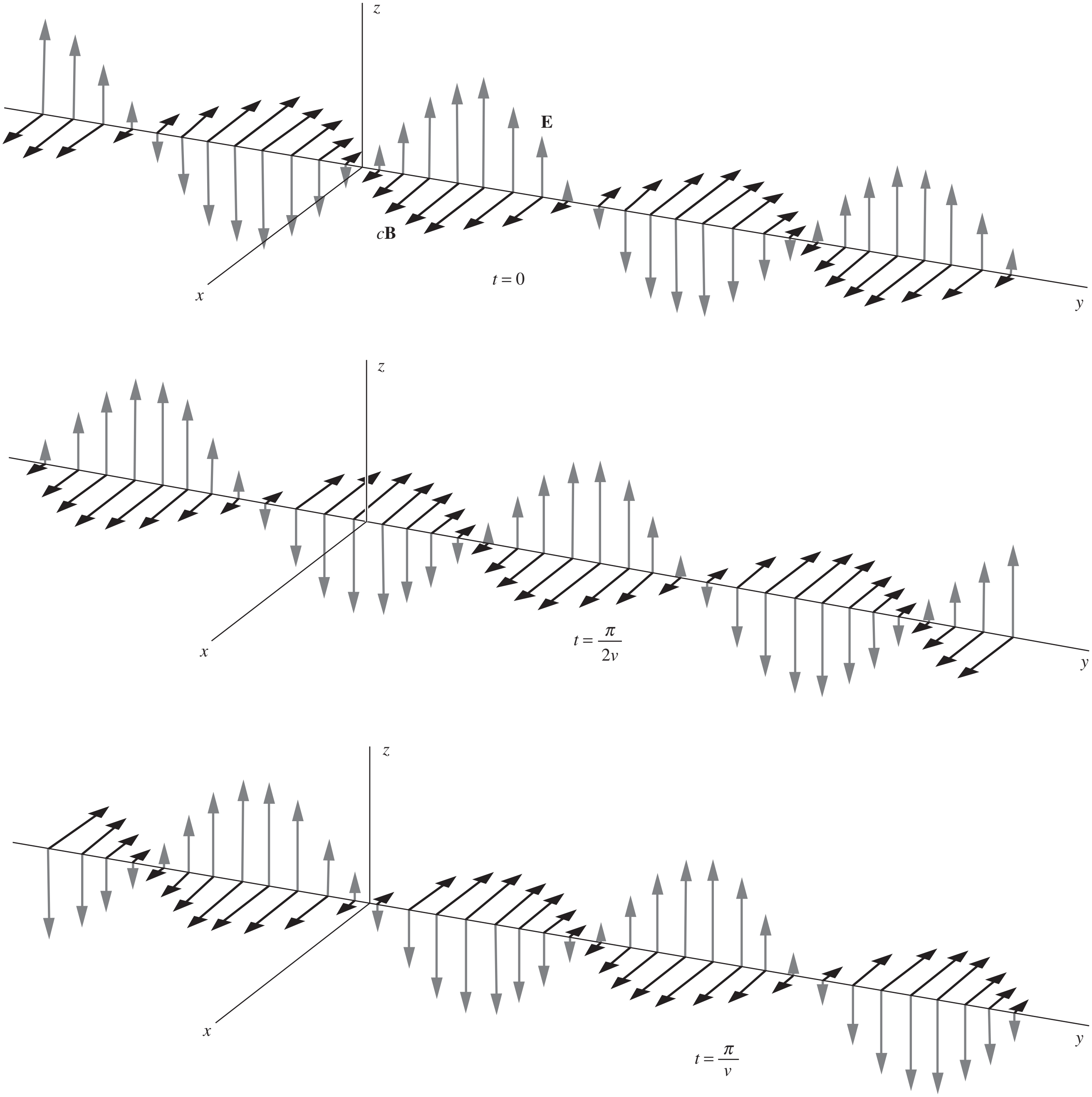


Figure 9.7.
The wave described by Eqs. (9.22) and (9.23) is shown at three different times. It is traveling to the right, in the positive y direction.

thanks to the particular form of the argument of the sine function in Eqs. (9.22) and (9.23); that argument, $y - vt$, has the same value at $y + \Delta y$ and $t + \Delta t$ as it had at y and t, providing $\Delta y = v\Delta t$. In other words, we have here a plane wave traveling with the constant speed v in the $\hat{\mathbf{y}}$ direction.

We'll show now that this electromagnetic field satisfies Maxwell's equations if certain conditions are met. It is easy to see that div $\mathbf{E}$ and div $\mathbf{B}$ are both zero for this field. The other derivatives involved are

$$\begin{aligned}
\operatorname{curl}\mathbf{E} &= \hat{\mathbf{x}}\frac{\partial E_z}{\partial y} = \hat{\mathbf{x}}E_0\cos(y - vt),\\
\frac{\partial \mathbf{E}}{\partial t} &= -v\hat{\mathbf{z}}E_0\cos(y - vt);\\
\operatorname{curl}\mathbf{B} &= -\hat{\mathbf{z}}\frac{\partial B_x}{\partial y} = -\hat{\mathbf{z}}B_0\cos(y - vt),\\
\frac{\partial \mathbf{B}}{\partial t} &= -v\hat{\mathbf{x}}B_0\cos(y - vt).
\end{aligned} \tag{9.24}$$

Substituting into the two "induction" equations of Eq. (9.18) and canceling the common factor, $\cos(y - vt)$, we find the conditions that must be satisfied are

$$E_0 = vB_0 \qquad \text{and} \qquad B_0 = \mu_0\epsilon_0 vE_0. \tag{9.25}$$

Together these require that

$$\boxed{v = \pm\frac{1}{\sqrt{\mu_0\epsilon_0}}} \qquad \text{and} \qquad \boxed{E_0 = \pm\frac{B_0}{\sqrt{\mu_0\epsilon_0}}} \tag{9.26}$$

Using $\mu_0\epsilon_0 = 1/c^2$ these relations become

$$\boxed{v = \pm c} \qquad \text{and} \qquad \boxed{E_0 = \pm cB_0} \tag{9.27}$$

We have now learned that our electromagnetic wave must have the following properties.

(1) *The field pattern travels with speed c*. In the case $v = -c$, it travels in the opposite, or $-\hat{\mathbf{y}}$, direction. When in 1862 Maxwell first arrived (by a more obscure route) at this result, the constant c in his equations expressed only a relation among electrical quantities as determined by experiments with capacitors, coils, and resistors. To be sure, the dimensions of this constant were those of velocity, but its connection with the actual speed of light had not yet been recognized. The speed of light had most recently been measured by Fizeau in 1857. Maxwell wrote, "The velocity of transverse undulations in our hypothetical medium, calculated from the electromagnetic experiments of MM. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from the optical experiments of M. Fizeau, that we can scarcely avoid the inference that *light consists in the transverse undulations of the same medium which*

is the cause of electric and magnetic phenomena." The italics are Maxwell's.

(2) *At every point in the wave at any instant of time, the electric field strength equals c times the magnetic field strength.* In our SI units, B is expressed in tesla and E in volts/meter. If the electric field strength is 1 volt/meter, the associated magnetic field strength is $1/(3 \cdot 10^8) = 3.33 \cdot 10^{-9}$ tesla. (In Gaussian units, the electric and magnetic field strengths are equal, with no need for the factor of c.)

(3) *The electric field and the magnetic field are perpendicular to one another and to the direction of travel, or propagation.* To be sure, we had already assumed this when we constructed our example, but it is not hard to show that it is a necessary condition, given that the fields do not depend on the coordinates perpendicular to the direction of propagation. Note that, if $v = -c$, which would make the direction of propagation $-\hat{\mathbf{y}}$, we must have $E_0 = -cB_0$. This preserves the handedness of the essential triad of directions, the direction of $\mathbf{E}$, the direction of $\mathbf{B}$, and the direction of propagation. We can describe this without reference to a particular coordinate frame as follows: the wave always travels in the direction of the vector $\mathbf{E} \times \mathbf{B}$.

Any plane electromagnetic wave in empty space has these three properties.

9.5 Other waveforms; superposition of waves

In the example we have just studied, the function $\sin(y - vt)$ was chosen merely for its simplicity. The "waviness" of the sinusoidal function has *nothing to do* with the essential property of wave motion, which is the propagation unchanged of a form or pattern – *any* pattern. It was not the nature of the function but the way y and t were combined in its argument that caused the pattern to propagate. If we replace the sine function by *any* other function, $f(y - vt)$, we obtain a pattern that travels with speed v in the $\hat{\mathbf{y}}$ direction. Moreover, Eq. (9.25) will apply as before (as you should check by working out the steps analogous to those in Eq. (9.24)), and our wave will have the three general properties just listed.

Here is another example, the plane electromagnetic wave pictured in Fig. 9.8, which is described mathematically as follows:

$$\mathbf{E} = \frac{E_0\hat{\mathbf{y}}}{1 + \dfrac{(x+ct)^2}{\ell^2}}, \qquad \mathbf{B} = \frac{-(E_0/c)\hat{\mathbf{z}}}{1 + \dfrac{(x+ct)^2}{\ell^2}}, \tag{9.28}$$

where ℓ is a fixed length that we have chosen as $\ell = 1$ foot for the purposes of drawing Fig. 9.8. (The speed of light is very nearly 1 foot/nanosecond.) This electromagnetic field satisfies Maxwell's equations, Eq. (9.18). It is a *plane* wave because nothing depends on y or z. It is traveling in the direction $-\hat{\mathbf{x}}$, as we recognize at once from the $+$ sign in the argument $x + ct$. That is indeed the direction of $\mathbf{E} \times \mathbf{B}$. In this

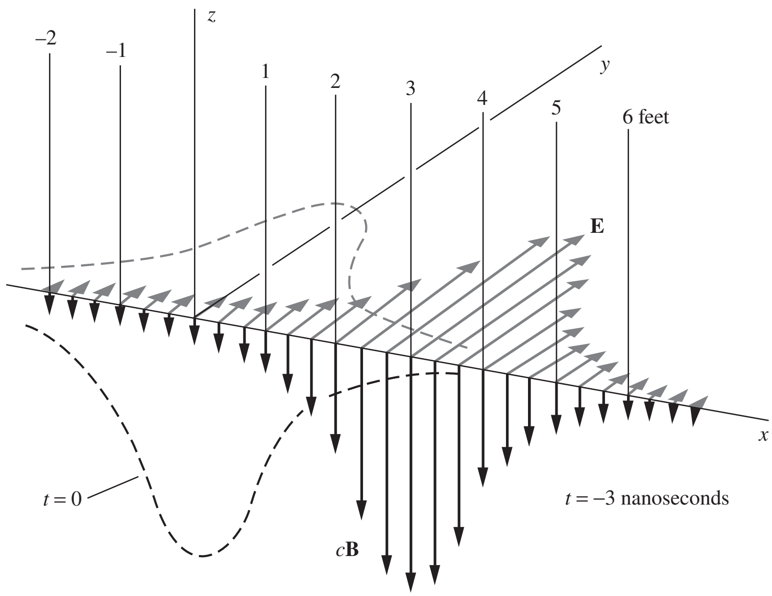


Figure 9.8.
The wave described by Eq. (9.28) is traveling in the negative x direction. It is shown 3 nanoseconds before its peak passes the origin.

wave nothing is oscillating or alternating; it is simply an electromagnetic pulse with long tails. At time $t = 0$, the maximum field strengths, $E = E_0$ (in volts/meter) and $B = E_0/c$ (which correctly has units of tesla) will be experienced by an observer at the origin, or at any other point on the yz plane. In Fig. 9.8 we have shown the field as it was at $t = -3$ nanoseconds, with the distances marked off in feet.

Maxwell's equations for **E** and **B** in empty space are linear. The superposition of two solutions is also a solution. Any number of electromagnetic waves can propagate through the same region without affecting one another. The field **E** at a space-time point is the vector sum of the electric fields of the individual waves, and the same goes for **B**.

Example (Standing wave) An important example is the superposition of two similar plane waves traveling in opposite directions. Consider a wave traveling in the $\hat{\mathbf{y}}$ direction, described by

$$\mathbf{E}_1 = \hat{\mathbf{z}}E_0 \sin\frac{2\pi}{\lambda}(y - ct), \qquad \mathbf{B}_1 = \hat{\mathbf{x}}\frac{E_0}{c}\sin\frac{2\pi}{\lambda}(y - ct). \tag{9.29}$$

This wave differs in only minor ways from the wave in Eqs. (9.22) and (9.23). We have introduced the wavelength λ of the periodic function, and we have used $B_0 = E_0/c$.

Now consider another wave:

$$\mathbf{E}_2 = \hat{\mathbf{z}}E_0 \sin\frac{2\pi}{\lambda}(y + ct), \qquad \mathbf{B}_2 = -\hat{\mathbf{x}}\frac{E_0}{c}\sin\frac{2\pi}{\lambda}(y + ct). \tag{9.30}$$

This is a wave with the same amplitude and wavelength, but propagating in the $-\hat{\mathbf{y}}$ direction. With the two waves both present, Maxwell's equations are still satisfied, the electric and magnetic fields now being

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \hat{\mathbf{z}}E_0 \left[\sin\left(\frac{2\pi y}{\lambda} - \frac{2\pi ct}{\lambda}\right) + \sin\left(\frac{2\pi y}{\lambda} + \frac{2\pi ct}{\lambda}\right)\right],$$
$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = \hat{\mathbf{x}}\frac{E_0}{c} \left[\sin\left(\frac{2\pi y}{\lambda} - \frac{2\pi ct}{\lambda}\right) - \sin\left(\frac{2\pi y}{\lambda} + \frac{2\pi ct}{\lambda}\right)\right]. \quad (9.31)$$

Remembering the formula for the sine of the sum of two angles, you can easily reduce Eq. (9.31) to

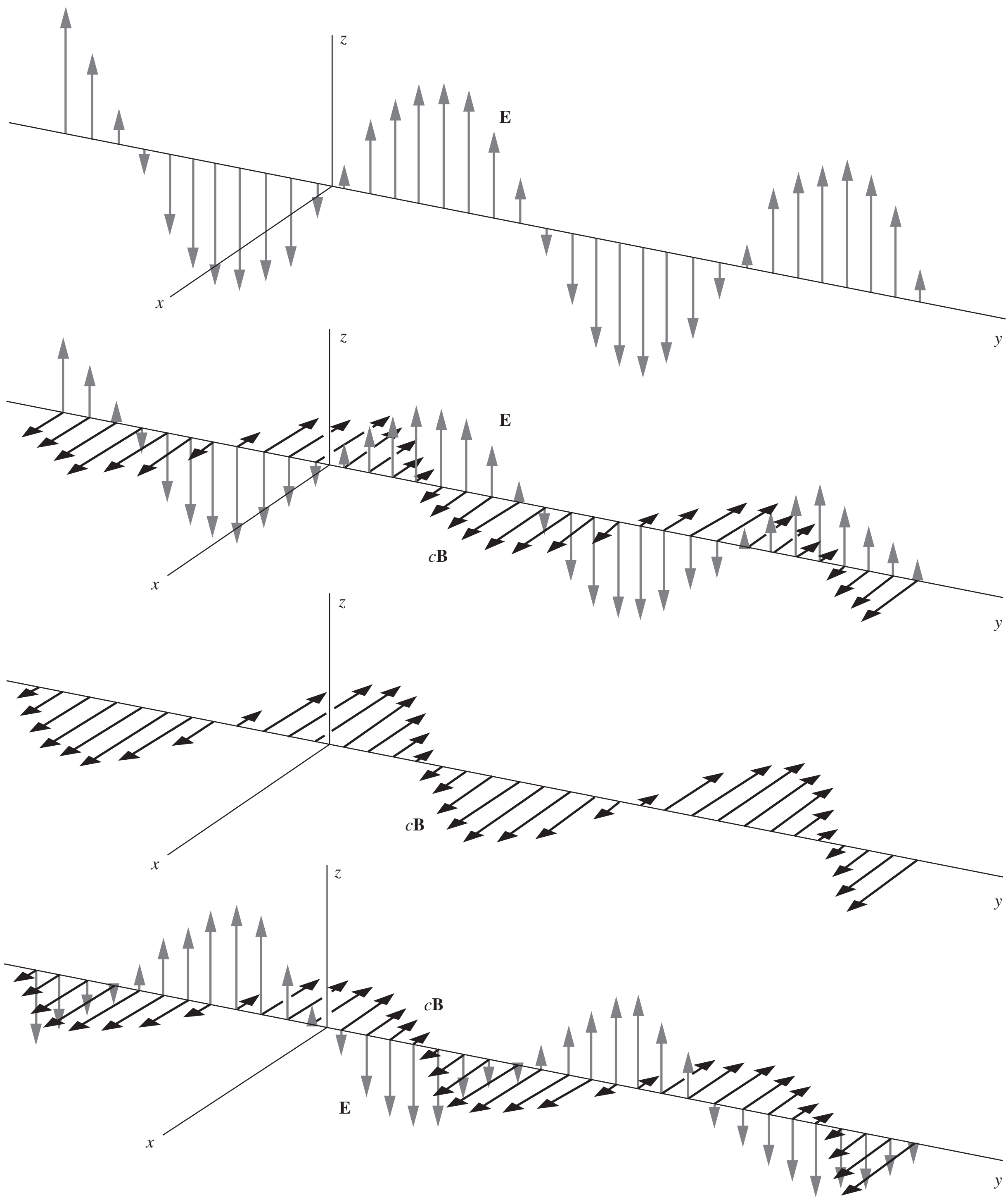
$$\mathbf{E} = 2\hat{\mathbf{z}}E_0 \sin\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda}, \qquad \mathbf{B} = -2\hat{\mathbf{x}}\frac{E_0}{c}\cos\frac{2\pi y}{\lambda}\sin\frac{2\pi ct}{\lambda}. \quad (9.32)$$

The field described by Eq. (9.32) is called a *standing wave*. Figure 9.9 suggests what it looks like at different times. The factor c/λ is the *frequency* (in time) with which the field oscillates at any position x, and $2\pi c/\lambda$ is the corresponding angular frequency. According to Eq. (9.32), whenever $2ct/\lambda$ equals an integer, which happens every half-period, we have $\sin 2\pi ct/\lambda = 0$, and the magnetic field $\mathbf{B}$ vanishes *everywhere*. On the other hand, whenever $2ct/\lambda$ equals an integer plus one-half, we have $\cos 2\pi ct/\lambda = 0$, and the electric field vanishes everywhere. The maxima of $\mathbf{B}$ and the maxima of $\mathbf{E}$ occur at different places as well as at different times. In contrast with the traveling wave, the standing wave has its electric and magnetic fields "out of step" in both space and time.

In the above standing wave, note that $\mathbf{E} = 0$ *at all times* on the plane $y = 0$ and on every other plane for which y equals an integral number of half-wavelengths. Imagine that we could cover the xz plane at $y = 0$ with a sheet of perfectly conducting metal. At the surface of a perfect conductor, the electric field component parallel to the surface must be zero – otherwise an infinite current would flow. That imposes a drastic *boundary condition* on any electromagnetic field in the surrounding space. But our standing wave, which is described by Eq. (9.32), *already satisfies* that condition, as well as satisfying Maxwell's equations in the entire space $y > 0$. Therefore it provides a ready-made solution to the problem of a plane electromagnetic wave reflected, at normal incidence, from a flat conducting mirror (see Fig. 9.10). The incident wave is described by Eq. (9.30), for $y > 0$, the reflected wave by Eq. (9.29). There is no field at all behind the mirror, or if there is, it has nothing to do with the field in front. Immediately in front of the mirror there is a magnetic field parallel to the surface, given by Eq. (9.32): $\mathbf{B} = -2\hat{\mathbf{x}}(E_0/c)\sin(2\pi ct/\lambda)$. The jump in $\mathbf{B}$ from this value in front of the conducting sheet to zero behind shows that an alternating current must be flowing in the sheet (see Section 6.6).

You could install a conducting sheet at any other plane where $\mathbf{E}$, as given by Eq. (9.32), is permanently zero, and thus trap an electromagnetic standing wave between two mirrors. That arrangement has many applications, including lasers. In fact, with an understanding of the

Figure 9.9 (see p. 444).
A standing wave, resulting from the superposition of a wave traveling in the positive y direction, Eq. (9.29), and a similar wave traveling in the negative y direction, Eq. (9.30). Beginning with the top figure, the fields are shown at four different times, separated successively by one-eighth of a full period.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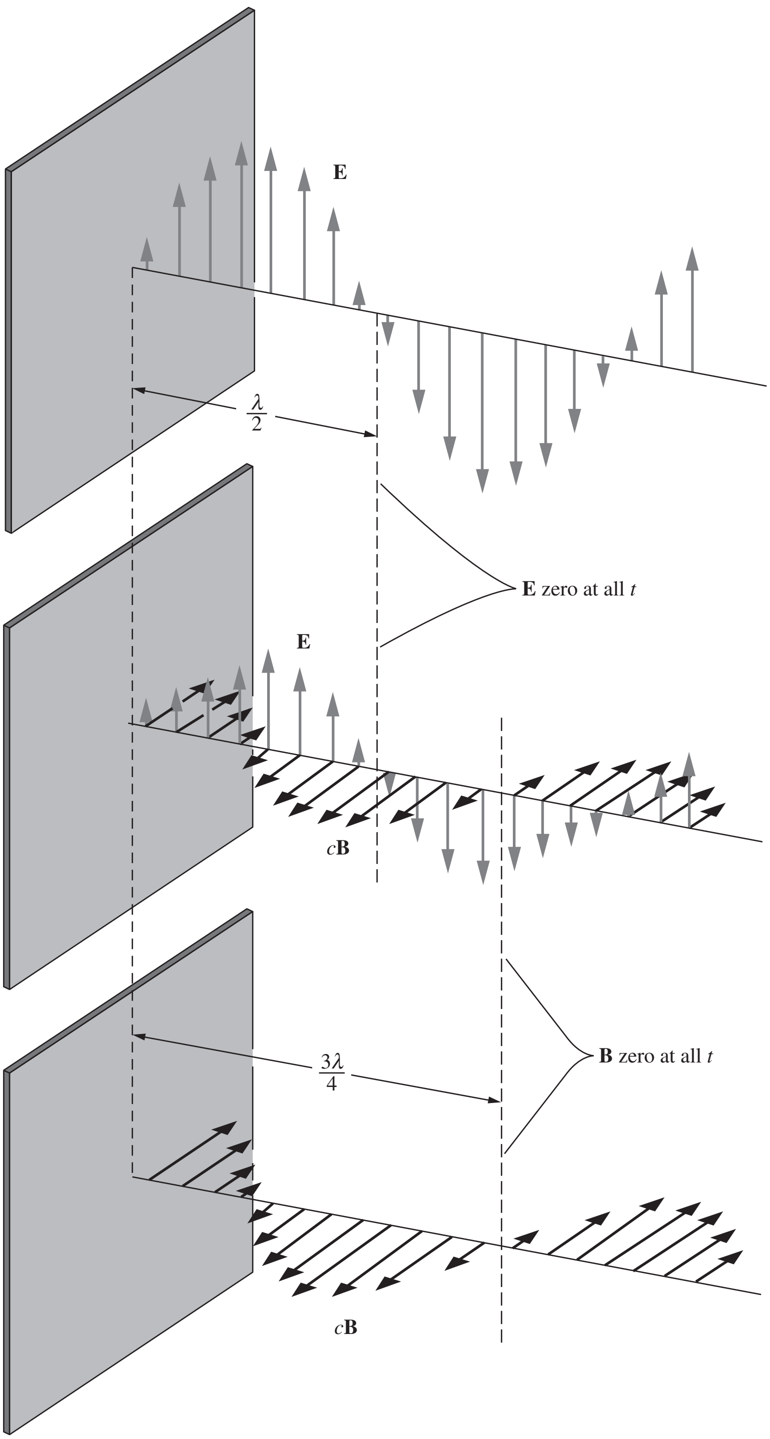

Figure 9.10.
A standing wave produced by reflection at a perfectly conducting sheet.

properties of the simple plane electromagnetic wave, you can analyze a surprisingly wide variety of electromagnetic devices, including interferometers, rectangular hollow wave guides, and strip lines.

9.2.2 ■ Monochromatic Plane Waves

For reasons discussed in Sect. 9.1.2, we may confine our attention to sinusoidal waves of frequency ω. Since different frequencies in the visible range correspond to different *colors*, such waves are called **monochromatic** (Table 9.1). Suppose,

[4]As Maxwell himself put it, "We can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena." See Ivan Tolstoy, *James Clerk Maxwell, A Biography* (Chicago: University of Chicago Press, 1983).

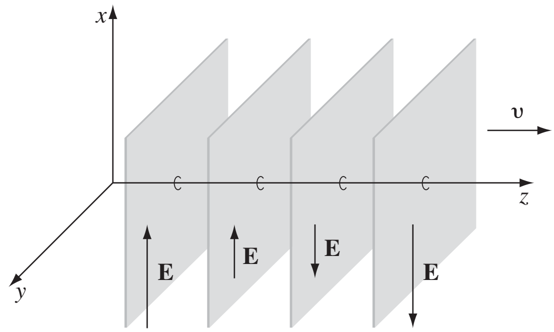

FIGURE 9.9

moreover, that the waves are traveling in the z direction and have no x or y dependence; these are called **plane waves**,[5] because the fields are uniform over every plane perpendicular to the direction of propagation (Fig. 9.9). We are interested, then, in fields of the form

$$\tilde{\mathbf{E}}(z, t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z, t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}, \tag{9.43}$$

where $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are the (complex) amplitudes (the *physical* fields, of course, are the real parts of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$), and $\omega = ck$.

Now, the wave equations for $\mathbf{E}$ and $\mathbf{B}$ (Eq. 9.41) were derived from Maxwell's equations. However, whereas every solution to Maxwell's equations (in empty space) must obey the wave equation, the converse is *not* true; Maxwell's equations impose extra constraints on $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$. In particular, since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$, it follows[6] that

$$(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0. \tag{9.44}$$

That is, *electromagnetic waves are transverse*: the electric and magnetic fields are perpendicular to the direction of propagation. Moreover, Faraday's law, $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, implies a relation between the electric and magnetic amplitudes, to wit:

$$-k(\tilde{E}_0)_y = \omega(\tilde{B}_0)_x, \quad k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y, \tag{9.45}$$

or, more compactly:

$$\tilde{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0). \tag{9.46}$$

[5]For a discussion of *spherical* waves, at this level, see J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory*, 3rd ed., Sect. 17-5 (Reading, MA: Addison-Wesley, 1979). Or work Prob. 9.35. Of course, over small enough regions *any* wave is essentially plane, as long as the wavelength is much less than the radius of the curvature of the wave front.

[6]Because the real part of $\tilde{\mathbf{E}}$ differs from the imaginary part only in the replacement of sine by cosine, if the former obeys Maxwell's equations, so does the latter, and hence $\tilde{\mathbf{E}}$ as well.

The Electromagnetic Spectrum		
Frequency (Hz)	Type	Wavelength (m)
10^{22}		10^{-13}
10^{21}	gamma rays	10^{-12}
10^{20}		10^{-11}
10^{19}		10^{-10}
10^{18}	x-rays	10^{-9}
10^{17}		10^{-8}
10^{16}	ultraviolet	10^{-7}
10^{15}	visible	10^{-6}
10^{14}	infrared	10^{-5}
10^{13}		10^{-4}
10^{12}		10^{-3}
10^{11}		10^{-2}
10^{10}	microwave	10^{-1}
10^{9}		1
10^{8}	TV, FM	10
10^{7}		10^{2}
10^{6}	AM	10^{3}
10^{5}		10^{4}
10^{4}	RF	10^{5}
10^{3}		10^{6}

The Visible Range		
Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

TABLE 9.1

Evidently, **E** and **B** are *in phase* and *mutually perpendicular;* their (real) amplitudes are related by

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0. \tag{9.47}$$

The fourth of Maxwell's equations, $\nabla \times \mathbf{B} = \mu_0 \epsilon_0 (\partial \mathbf{E}/\partial t)$, does not yield an independent condition; it simply reproduces Eq. 9.45.

Example 9.2. If **E** points in the x direction, then **B** points in the y direction (Eq. 9.46):

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}(z,t) = \frac{1}{c}\tilde{E}_0 e^{i(kz-\omega t)}\hat{\mathbf{y}},$$

or (taking the real part)

$$\boxed{\mathbf{E}(z,t) = E_0\cos(kz-\omega t+\delta)\,\hat{\mathbf{x}}, \quad \mathbf{B}(z,t) = \frac{1}{c}E_0\cos(kz-\omega t+\delta)\,\hat{\mathbf{y}}.} \tag{9.48}$$

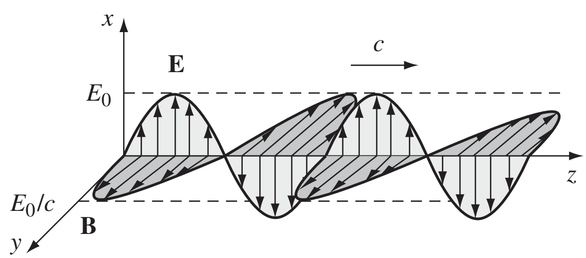

FIGURE 9.10

This is the paradigm for a monochromatic plane wave (see Fig. 9.10). The wave as a whole is said to be polarized in the x direction (by convention, we use the direction of **E** to specify the polarization of an electromagnetic wave).

There is nothing special about the z direction, of course—we can easily generalize to monochromatic plane waves traveling in an arbitrary direction. The notation is facilitated by the introduction of the **propagation** (or **wave**) **vector**, **k**, pointing in the direction of propagation, whose magnitude is the wave number k. The scalar product $\mathbf{k}\cdot\mathbf{r}$ is the appropriate generalization of kz (Fig. 9.11), so

$$\boxed{\begin{aligned} \tilde{\mathbf{E}}(\mathbf{r},t) &= \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{n}}, \\ \tilde{\mathbf{B}}(\mathbf{r},t) &= \frac{1}{c}\tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}}\times\hat{\mathbf{n}}) = \frac{1}{c}\hat{\mathbf{k}}\times\tilde{\mathbf{E}}, \end{aligned}} \tag{9.49}$$

where $\hat{\mathbf{n}}$ is the polarization vector. Because **E** is transverse,

$$\hat{\mathbf{n}}\cdot\hat{\mathbf{k}} = 0. \tag{9.50}$$

(The transversality of **B** follows automatically from Eq. 9.49.) The actual (real) electric and magnetic fields in a monochromatic plane wave with propagation vector **k** and polarization $\hat{\mathbf{n}}$ are

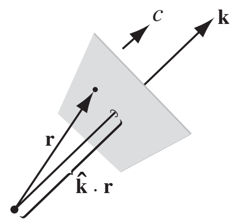

FIGURE 9.11

$$\mathbf{E}(\mathbf{r}, t) = E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)\, \hat{\mathbf{n}}, \tag{9.51}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)(\hat{\mathbf{k}} \times \hat{\mathbf{n}}). \tag{9.52}$$

Problem 9.9 Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero that is (a) traveling in the negative x direction and polarized in the z direction; (b) traveling in the direction from the origin to the point (1, 1, 1), with polarization parallel to the xz plane. In each case, sketch the wave, and give the explicit Cartesian components of $\mathbf{k}$ and $\hat{\mathbf{n}}$.

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